PRIMARY IDEALS AND VALUATION IDEALS. II

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Introduction. Let D be an integral domain with identity. In [2], Gilmer and Ohm considered the problem of characterizing domains D such that the set $\mathscr{Q}(D)$ of primary ideals of D is a subset of the set $\mathscr{V}(D)$ of valuation ideals of D. If the ascending chain condition (a.c.c.) for prime ideals holds in D, then $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$ if and only if D is a Prüfer domain; a domain in which primary ideals are valuation ideals need not be Prüfer if the assumption concerning the a.c.c. for prime ideals is dropped (see Theorem 3.8 and §5 of [2]). In case a.c.c. for prime ideals does not hold in D, Gilmer and Ohm left open the question as to when primary ideals of D are valuation ideals.

In [1], Gilmer showed that the question as to whether $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$ or not is closely related to the structure of the set of prime ideals of D. Before mentioning these results and their relation to this paper, we introduce some terminology. Let R be a commutative ring with identity, let P be a prime ideal of R, and let $\{Q_\alpha\}$ be the set of P-primary ideals of R. We consider the following conditions:

I. $\{Q_\alpha\}$ is linearly ordered under $\subseteq$.

II. $M=\bigcap_\alpha Q_\alpha$ is a prime ideal.

III. There are no prime ideals of R properly between M and P.

IV. If P_1 is any prime ideal of R properly contained in P, then $P_1\subseteq M$.

Following [1], we say P is an *S-ideal* if I, II, and IV hold. It is clear that IV implies III. If I–III hold, we say P is a *weak S-ideal.* R is an *S-ring* if each prime ideal of R is an S-ideal; *weak S-ring* is defined analogously. Corollary 2.4 of [1] shows that if D is an S-domain, then $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$. The proof of Corollary 2.4 does, in fact, show that in a weak S-domain primary ideals are valuation ideals. The status of the converse of Corollary 2.4 was considered in [1], but was not determined.

In §3 we prove that for P prime in D, each P-primary ideal is a valuation ideal if and only if conditions I and II hold. We thereby obtain what we feel is a satisfactory characterization of domains in which primary ideals are valuation ideals. To resolve the questions of whether the condition $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$ implies D is an S-domain or a weak S-domain we need only determine in the global case whether condition IV or condition III depends upon I and II. These questions are answered in §5. Example 5.8 shows III does not depend on I and II and hence IV does not

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depend on I and II. Example 5.9 shows IV is independent of I–III—that is, a weak S-domain need not be an S-domain. We begin in §2 by considering a prime ideal P of a commutative ring with identity such that the set of P-primary ideals is chained under $\subseteq$.

All rings considered in this paper are assumed to be commutative and we assume that each integral domain considered has an identity. The terminology used is that of Zariski-Samuel [6], [7]. We shall make frequent use of the material from [6] on quotient rings of, in our case, an integral domain. We shall also use the fact that in considering v-ideals of a domain D, where v is a valuation nonnegative on D, there is no loss of generality in assuming that D and the valuation ring of v have the same quotient field [7, p. 340]. While we feel that a reader can understand most of the results of this paper without having read [2], it is unlikely that he can appreciate its results independent of [1] and [2].

2. **Linearly ordered systems of primary ideals.** Throughout this section P denotes a prime ideal of R, a commutative ring with identity, and $\{Q_\alpha\}$ denotes the set of P-primary ideals. We shall assume that the set $\{Q_\alpha\}$ is linearly ordered under $\subseteq$; we denote by M the intersection of the Q_α's.

2.1. LEMMA. *If $(Q_\beta\}$ is a subset of $\{Q_\alpha\}$ such that $B=\bigcap_\beta Q_\beta \supset M$, then $\sqrt{B}=P$. Equivalently, if $\sqrt{B}\subset P$, then $B=M$.*

Proof. Since $B\supset M$, there exists a P-primary ideal Q such that $B\nsubseteq Q$. Thus $Q_\beta \nsubseteq Q$ for any β so that $Q\subseteq Q_\beta$ for each β, and consequently, $Q\subseteq \bigcap_\beta Q_\beta=B$. Hence $P=\sqrt{Q}\subseteq\sqrt{B}\subseteq P$.

We shall find it convenient in the results immediately following to consider the case when P is maximal in R. We are able to relate these results to similar questions in the general case by observing that in the quotient ring R_P, P^e is maximal and $\{Q_\alpha^e\}$, the set of P^e-primary ideals of R_P, is linearly ordered under $\subseteq$; here "e" denotes extension with respect to R_P [6, p. 218]. Then we use known results concerning the relationship between the ideal theory of R and that of R_P [6, pp. 223–233].

2.2. PROPOSITION. *If P is maximal in R, then for any P-primary ideal Q, either $PQ=Q$ or $PQ\subset Q$ and each P-primary ideal properly contained in Q is contained in PQ.*

Proof. We suppose $PQ\subset Q$. Then Q/PQ is an R-module and P is contained in the annihilator of this module over R. Hence the structure of Q/PQ as an R-module is the same as its structure as an R/P-module—that is, as a vector space over R/P. The submodules of Q/PQ correspond to the ideals A of R such that $PQ\subseteq A\subseteq Q$. Any such ideal A has radical P and must therefore be P-primary since P is maximal in R [6, p. 153]. Thus, by hypothesis on $\{Q_\alpha\}$, the subspaces of the vector space Q/PQ over R/P are linearly ordered so that Q/PQ is one dimensional. It follows that there are no ideals of R properly between PQ and Q. Since each P-primary ideal Q' properly contained in Q compares with PQ under $\subseteq$, it follows that each such Q' must be contained in PQ. Q.E.D.

Before proceeding further we introduce some notation. If $x \in R-M$ we denote by B_x the intersection of all P-primary ideals which contain x (if $x \notin P$, then $B_x = R$). If $x \in P-M$, Lemma 2.1 shows that B_x is an ideal of R having radical P. Hence if P is maximal and if $x \in P-M$, then B_x is the smallest P-primary ideal of R containing x. For P maximal we have for any $x \in R-M$, $B_x = \bigcap_\alpha (Q_\alpha + (x))$.

2.3. Lemma. *If P is maximal in R and if $x, y \in R-M$, then $B_xB_y \subseteq \bigcap_\alpha (Q_\alpha + (xy))$. If $xy \notin M$, then $B_xB_y = B_{xy}$.*

Proof. If $s \in B_x$ and $t \in B_y$, then for any α we have $s \in Q_\alpha + (x)$ and $t \in Q_\alpha + (y)$ so that $st \in (Q_\alpha + (x))(Q_\alpha + (y)) \subseteq Q_\alpha + (xy)$. Hence $st \in \bigcap_\alpha (Q_\alpha + (xy))$ and

$$B_xB_y \subseteq \bigcap\nolimits_\alpha (Q_\alpha + (xy))$$

as we wished to show.

If $xy \notin M$ then we have just shown that $B_xB_y \subseteq B_{xy}$. However $\sqrt{(B_xB_y)} = P$ so that B_xB_y is a P-primary ideal containing xy. Hence $B_{xy} \subseteq B_xB_y$. Q.E.D.

2.4. Proposition. *If P is maximal in R and if M does not have radical P, then M is a prime ideal.*

Proof. Let $x, y \in R-M$. We show $xy \in R-M$. If $x, y \in R-P$ it is clear that $xy \in R-M$. If $xy \in P$, Lemma 2.3 shows that $B_xB_y \subseteq \bigcap_\alpha (Q_\alpha + (xy)) \subseteq P$ so that $\sqrt{(\bigcap_\alpha (Q_\alpha + (xy)))} = P$. Because $\sqrt{M} = \sqrt{(\bigcap_\alpha Q_\alpha)} \subset P$ it follows that for some α, $xy \notin Q_\alpha$. Therefore $xy \notin M$ and M is a prime ideal. Q.E.D.

2.5. Corollary. *If M does not have radical P, M is a prime ideal.*

Proof. If e and c denote extension and contraction with respect to R and R_P (see [6, p. 218]) we first show that $M = M^{ec}$. $\subseteq$ is clear. If $x \in M^{ec}$ there is an element $y \in R-P$ such that $xy \in M$. Thus $xy \in Q_\alpha$ for each α so that $x \in Q_\alpha$ for each α since Q_α is P-primary. Hence $x \in \bigcap_\alpha Q_\alpha = M$ and we have established that $M = M^{ec}$. Thus if $t \in P - \sqrt{M}$, then for any positive integer n, $t^n \notin M$. It follows that $t^n \notin M^{ec}$ so that $\sqrt{M^e} \subset P^e$. But it is easily seen that $M^e = (\bigcap_\alpha Q_\alpha)^e = \bigcap_\alpha Q_\alpha^e$, and $\{Q_\alpha^e\}$ is the set of P^e-primary ideals of R_P. Applying Proposition 2.4 to the ring R_P it follows that M^e is prime in R_P. Hence $M = M^{ec}$ is prime in R. Q.E.D.

2.6. Proposition. *If P is maximal in R and if the powers of P properly descend, then $\{P^n\}_{n=1}^\infty$ is the set of P-primary ideals and $M = \bigcap_{n=1}^\infty P^n$ is a prime ideal.*

Proof. By Proposition 2.2 each P-primary ideal properly contained in P^n, for any positive integer n, is contained in P^{n+1}. We use this fact to show $A = \bigcap_{n=1}^\infty P^n$ is a prime ideal. If $x, y \in R-A$ we may choose integers k, t such that $x \in P^k - P^{k+1}$ and $y \in P^t - P^{t+1}$, where $P^0 = R$. Then $P^k = P^{k+1} + (x)$ and $P^t = P^{t+1} + (y)$ since there are no ideals of R properly between P^n and P^{n+1} for any nonnegative n. Hence $P^{k+t} = (P^{k+1} + (x))(P^{t+1} + (y)) \subseteq P^{k+t+1} + (xy)$. Since $P^{k+t} \supset P^{k+t+1}$, it follows that $xy \notin P^{k+t+1}$ so that $xy \notin A$ and A is prime.

Now if Q is P-primary, $Q \nsubseteq A$ so $Q \nsubseteq P^k$ for some k. Therefore $Q \supseteq P^k$ and if t is such that $Q \supseteq P^t$ but $Q \nsupseteq P^{t-1}$, we must have $Q = P^t$. This completes the proof.

As with Corollary 2.5, Corollary 2.7 follows immediately from Proposition 2.6 by passage to the quotient ring R_P. We therefore omit the proof.

2.7. COROLLARY. *If the symbolic powers of P properly descend, then $\{P^{(n)}\}_{n=1}^{\infty}$ is the set of P-primary ideals and $M = \bigcap_{n=1}^{\infty} P^{(n)}$ is prime.*

3. **A characterization of domains for which $\mathscr{Q}(D) \subseteq \mathscr{V}(D)$.** Let P be a prime ideal of D, a domain with identity; let $\{Q_\alpha\}$ be the set of P-primary ideals. Our purpose in this section is to establish the validity of the following statement $(*)$:

$(*)$: *In order that each Q_α be a valuation ideal it is necessary and sufficient that $\{Q_\alpha\}$ be linearly ordered under $\subseteq$ and that $\bigcap_\alpha Q_\alpha$ be a prime ideal.*

We first achieve a reduction to the case when D is quasi-local via Theorem 3.1.

3.1. THEOREM. *If $(*)$ holds when D is quasi-local, then $(*)$ is true in any domain with identity.*

Proof. D_P is a quasi-local domain with maximal ideal PD_P, and $\{Q_\alpha D_P\}$ is the set of PD_P-primary ideals. Since $Q_\alpha D_P \cap D = Q_\alpha$, it follows that if $Q_\alpha D_P$ is a valuation ideal, then Q_α is a valuation ideal. Corollary 2.6 of [2] shows that the converse is also valid. It is clear that $\{Q_\alpha D_P\}$ is linearly ordered under $\subseteq$ if and only if $\{Q_\alpha\}$ is linearly ordered under $\subseteq$. Finally, the proof of Corollary 2.5 shows that if $\bigcap_\alpha Q_\alpha$ is prime then $\bigcap_\alpha Q_\alpha D_P$ is also prime, and the converse is obvious. All these observations establish Theorem 3.1.

We are now able to establish in Theorem 3.2 necessity of the conditions given in $(*)$. The proof of Theorem 3.2 uses the following result:

If R is a commutative ring such that the set of principal ideals of R is linearly ordered under $\subseteq$, then the set of ideals of R is linearly ordered under $\subseteq$.

3.2. THEOREM. *Suppose J is a quasi-local domain with maximal ideal M. If M-primary ideals of J are valuation ideals, the set of M-primary ideals is linearly ordered under $\subseteq$ and the intersection of the set of M-primary ideals is a prime ideal.*

Proof. Suppose Q is M-primary. We first show that for $x, y \in J$,

$$Q+(x) \subseteq Q+(y) \qquad \text{or} \qquad Q+(y) \subseteq Q+(x).$$

This is clear unless $x, y \in M$. In this case $Q^2+(xy)$ has radical M, is therefore M-primary, and hence is a valuation ideal. Thus $x^2 \in Q^2+(xy)$ or $y^2 \in Q^2+(xy)$—say $x^2 \in Q^2+(xy)$; $x^2 = q+rxy$ where $q \in Q^2$ and $r \in J$. It follows that $x(x-ry) = q \in Q^2$. But Q is a v-ideal for some valuation v. Hence Lemma 2.8 of [2] applies to show that either $x \in Q$ or $x-ry \in Q$. Consequently $Q+(x) = Q \subseteq Q+(y)$ or $x = (x-ry)+ry \in Q+(y)$ and again $Q+(x) \subseteq Q+(y)$. It then follows that the set of principal ideals of D/Q are linearly ordered under $\subseteq$ so that the ideals of D/Q are linearly ordered under $\subseteq$. But this implies that the ideals of D which contain

Q form a chain—this statement holds for any M-primary ideal Q. In particular, if Q_1 and Q_2 are M-primary, $Q_1 \cap Q_2$ is also M-primary and therefore $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. We have shown that the set of M-primary ideals is linearly ordered under $\subseteq$. That the intersection of all M-primary ideals is a prime ideal is the content of Proposition 2.14 of [2].

We still wish to show that if $\{Q_\alpha\}$ is the set of P-primary ideals of a quasi-local domain D with maximal ideal P, if $\{Q_\alpha\}$ is chained under $\subseteq$, and if $\bigcap_\alpha Q_\alpha = M$ is a prime ideal, then each Q_α is a valuation ideal. Lemmas 3.3 and 3.4 allow us to reduce this problem to the case when $M=(0)$.

3.3. Lemma. *Let R be a subring of the ring S, let B be an ideal of S and let $A_0 = B \cap R$. Let $\phi: S \to S/B$ be the canonical homomorphism. We identify R/A_0 with $\phi(R) = (R+B)/B \simeq R/(R \cap B)$. Let C be an ideal of R containing A_0, $\bar{T}$ be a ring between R/A_0 and S/B such that some ideal $\bar{N}$ of $\bar{T}$ lies over C/A_0: $\bar{N} \cap (R/A_0) = C/A_0$. If $T = \phi^{-1}(\bar{T})$, T is a ring between R and S and $N = \phi^{-1}(\bar{N})$ is an ideal of T lying over C.*

Proof. The proof is immediate and will be omitted.

3.4. Lemma. *Let A be an ideal of a domain D and let P_1 be a prime ideal of D contained in A such that A/P_1 is a valuation ideal of D/P_1. Then A is a valuation ideal of D.*

Proof. Let V be a valuation ring between D and its quotient field such that V has center P_1 on D. If M is the maximal ideal of V, then $V/M = \bar{k}$ contains D/P_1 to within isomorphsm and therefore contains the quotient field $\bar{k}_1$ of D/P_1. By hypothesis there is a valuation ring $\bar{V}_1$ and an ideal $\bar{A}_1$ of $\bar{V}_1$ such that

$$D/P_1 \subseteq \bar{V}_1 \subseteq \bar{k}_1 \qquad \text{and} \qquad \bar{A}_1 \cap (D/P_1) = A/P_1.$$

The valuation ring $\bar{V}_1$ has an extension to a valuation ring $\bar{V}$ with quotient field $\bar{k}$. We show that $\bar{A}_1\bar{V} \cap \bar{V}_1 = \bar{A}_1$. $\supseteq$ always holds. We consider a nonzero element $a = \sum_{i=1}^n a_i t_i \in \bar{A}_1 \bar{V} \cap \bar{V}_1$, where $a_i \in \bar{A}_1$ and $t_i \in \bar{V}$. The ideal of $\bar{V}$ generated by $\{a_1, \ldots, a_n\}$ is principal and is generated by some a_i—say by a_1. Then $a = a_1 t$ for some $t \in \bar{V}$. Hence $t = a/a_1 \in \bar{V} \cap \bar{k}_1 = \bar{V}_1$, implying that $a \in a_1 \bar{V}_1 \subseteq \bar{A}_1$. This proves that $\bar{A}_1 \bar{V} \cap \bar{V}_1 = \bar{A}_1$ so that the ideal $\bar{A}_1 \bar{V}$ of $\bar{V}$ lies over A/P_1 in D/P_1.

We now apply Lemma 3.3 with $R = D$, $S = V$, $B = M$, $C = A$, $\bar{T} = \bar{V}$, and $\bar{N} = \bar{A}_1 \bar{V}$. We conclude that if ϕ is the canonical homomorphism from V onto V/M, then $\phi^{-1}(\bar{V})$ is a domain between D and V containing the ideal $\phi^{-1}(\bar{A}_1\bar{V})$ lying over A. But V and $\bar{V}$ are valuation rings so that $\phi^{-1}(\bar{V})$ is also a valuation ring [4, p. 34, Result 11.4]. Hence A is a valuation ideal. Q.E.D.

Lemma 3.4 shows that in proving (*) there is no loss of generality in assuming $\bigcap_\alpha Q_\alpha = (0)$, for if we show Q_α/M is a valuation ideal of the domain D/M, where $M = \bigcap_\alpha Q_\alpha$, then Lemma 3.4 shows that Q_α is also a valuation ideal. Also, $\{Q_\alpha/M\}$ is the set of P/M-primary ideals of D/M, this set is chained under $\subseteq$, and

$\bigcap_\alpha (Q_\alpha/M)=(\bar{0})$. We therefore introduce the following notation for the remainder of this section: J denotes a quasi-local domain with maximal ideal P; $\{Q_\alpha\}$ is the set of P-primary ideals. We assume $\{Q_\alpha\}$ is linearly ordered under $\subseteq$ and that $\bigcap_\alpha Q_\alpha=(0)$. As in §1 we define, for $x \in J$, $x\neq 0$, $B_x=\bigcap_\alpha (Q_\alpha+(x))$. Our object is to prove Theorem 3.5, from which the validity of (*) will follow.

3.5. THEOREM. *Each Q_α is a valuation ideal.*

Our proof will require the following lemma.

3.6. LEMMA. *For $a, b, c, d \in J$ if $a/b=c/d$, then $B_a\subseteq B_b$ if and only if $B_c\subseteq B_d$.*

Proof. The conclusion being symmetric, we need only prove that $B_a\subseteq B_b$ implies $B_c\subseteq B_d$. Since B_c and B_d are P-primary, $B_c\subseteq B_d$ or $B_d\subseteq B_c$. We show that if $B_d\subseteq B_c$, then $B_d=B_c$. Thus let Q be a P-primary ideal such that $ad\notin Q$. Since $B_a\subseteq B_b$, $a\in Q+(b)$—say $a=q_1+r_1b$ where $q_1\in Q$ and $r_1\in J$. By assumption $ad\notin Q$ so that $d\notin Q$. This implies that $Q\subset B_d$, and hence that $Q+(d)\subseteq B_d$. $Q+(d)$ is P-primary and contains d, however, so that equality holds: $B_d=Q+(d)$. Also, $B_d\subseteq B_c\subseteq Q+(c)$ so that $d=q_2+r_2c$ for some $q_2\in Q$, $r_2\in J$. Then $bc=ad=(q_1+r_1b)\times(q_2+r_2c)\equiv r_1r_2bc$ modulo Q. Thus $(1-r_1r_2)bc\in Q$. Because $bc\notin Q$ this implies $1-r_1r_2$ is a nonunit of J, implying that r_1 and r_2 are units of J. In particular $c=r_2^{-1}(d-q_2)\in Q+(d)=B_d$, $B_c\subseteq B_d$, and our proof is complete. Q.E.D.

Proof of Theorem 3.5. We define $V=\{a/b \mid a, b\in J, b\neq 0, \text{ and } B_a\subseteq B_b\}$. We show V is a valuation ring between J and its quotient field K such that $Q_\alpha V\cap J=Q_\alpha$ for each α. First, Lemma 3.6 shows that V is a well-defined subset of K, and it is clear that $J\subseteq V$. We show V is a valuation ring. If $a/b, c/d\in V$ we have $B_a\subseteq B_b$ and $B_c\subseteq B_d$. By Lemma 2.3, $B_{ac}=B_aB_c\subseteq B_bB_c\subseteq B_bB_d=B_{bd}$. Thus $ac/bd=(a/b)(c/d)\in V$. Similarly, $B_{ad}\subseteq B_{bd}$, $B_{bc}\subseteq B_{bd}$, and hence $B_{ad-bc}\subseteq B_{ad}+B_{bc}\subseteq B_{bd}$, implying $(ad-bc)/bd=(a/b)-(c/d)\in V$. Finally, the linear ordering on the set of P-primary ideals implies for $a, b\in J$ that $B_a\subseteq B_b$ or $B_b\subseteq B_a$. Thus a/b or b/a is in V, and V is a valuation ring.

For any α we have $Q_\alpha\subseteq Q_\alpha V\cap J$. And any nonzero element x of $Q_\alpha V\cap J$ is of the form qt for some $q\in Q_\alpha$, $t\in V$. We have $t=x/q$ with $x, q\in J$ so that $B_x\subseteq B_q\subseteq Q_\alpha$. Hence $x\in Q_\alpha$ and $Q_\alpha V\cap J=Q_\alpha$ as we wished to show.

We remark that we have actually established the validity of the following Corollary 3.7. The question of the status of this result was raised in [2].

3.7. COROLLARY. *Suppose P is prime in the domain D and that each P-primary ideal is a valuation ideal. Then there exists a valuation v, nonnegative on D, such that each P-primary ideal is a v-ideal.*

4. **Further properties of domains for which $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$.** We have shown in §3 that if $\{Q_\alpha\}$, the set of P-primary ideals of a domain D, is chained under $\subseteq$ and if $\bigcap_\alpha Q_\alpha$ is a prime ideal of D, then each Q_α is a valuation ideal. In fact, there is a single valuation v such that each Q_α is a v-ideal. We investigate in this section the

uniqueness of such a valuation v, the dependence of the condition "$\bigcap_\alpha Q_\alpha$ is prime" on the condition "$\{Q_\alpha\}$ is chained under $\subseteq$", and some questions raised in [1] and [2] concerning the structure of the set of prime ideals of a domain in which primary ideals are valuation ideals.

Before proving Theorem 4.1 we need some additional terminology. Following [1, p. 252] we say that a prime ideal P of a commutative ring R is *branched* provided there exists $Q \neq P$ such that Q is P-primary; otherwise we say P is *unbranched.* If R is a Prüfer domain—in particular if R is a valuation ring—then P is branched if and only if P properly contains the union of the chain of all prime ideals properly contained in P [1, p. 252, Lemma 3.4]. If D is a domain with quotient field K, if V is a valuation ring between D and K having maximal ideal M, and if P is the center of V on D, we say V has *absolute center* P on D provided M is the only prime ideal of V lying over P.

4.1. THEOREM. *Let P be a branched prime ideal of a domain D with quotient field K such that the set $\{Q_\alpha\}$ of P-primary ideals is linearly ordered under $\subseteq$. Then these statements are equivalent*:

(a) *$\bigcap_\alpha Q_\alpha$ is a prime ideal.*

(b) *Each Q_α is a valuation ideal.*

(c) *Some $Q_\alpha \neq P$ is a valuation ideal.*

(d) *There exists $Q_\alpha \neq P$ and a valuation ring V between D and K such that $Q_\alpha V \neq PV$.*

(e) *P is the absolute center of some valuation ring with branched maximal ideal.*

Proof. (a)$\leftrightarrow$(b): By §3.

(b)$\rightarrow$(c): Obvious.

(c)$\rightarrow$(d): Clear.

(d)$\rightarrow$(e): Let Q_α and V be as in (d). Then we have $Q_\alpha V \subset PV \subseteq M = (PV)^{1/2}$. Since V is a valuation ring, M is a prime ideal; also $M = (Q_\alpha V)^{1/2}$ where $Q_\alpha V \subset M$. It follows that M is branched [1, p. 248, Lemma 1.6]. Therefore $W = V_M$ is a valuation ring having absolute center P on D and such that the maximal ideal $M = MV_M$ of W is branched.

(e)$\rightarrow$(a): Let V be a valuation ring between D and K having absolute center P on D and the maximal ideal M of V is branched. Then if $\{A_\beta\}$ is the set of M-primary ideals, $M' = \bigcap_\beta A_\beta$ is a prime ideal of V properly contained in M [2, p. 241, Proposition 2.14]. Because V has absolute center P on D, it follows that $M' \cap D = (\bigcap_\beta A_\beta) \cap D = \bigcap_\beta (A_\beta \cap D) = P'$ is a prime ideal of D properly contained in P. Hence $\{A_\beta \cap D\}$ is a collection of P-primary ideals whose intersection does not have radical P. By Lemma 2.1, $\bigcap_\beta (A_\beta \cap D) = \bigcap_\alpha Q_\alpha$ and (a) holds. Q.E.D.

4.2. COROLLARY. *If the ascending chain condition for prime ideals holds in the domain D and if P-primary ideals of D are linearly ordered for each prime ideal P of D, then D is a Prüfer domain.*

Proof. Let M be a proper prime ideal of D. Since the a.c.c. for prime ideals holds in D, there is a prime ideal $M_1 \subset M$ such that there are no prime ideals of D properly between M_1 and M. By Lemma 3.2 of [2], M_1 is an intersection of a collection of M-primary ideals. By Lemma 2.1, M_1 is therefore the intersection of all M-primary ideals. By Theorem 4.1, primary ideals of D are valuation ideals. Theorem 3.8 of [2] then shows D is Prüfer.

In a preprint of this paper we left as an open question whether the hypothesis that $\{Q_\alpha\}$ is linearly ordered in Theorem 4.1 implies that $\bigcap_\alpha Q_\alpha$ is prime. The following example, communicated to us by M. Nagata and T. Kikuchi, shows that the answer to this question is negative. Let V be a valuation ring with unbranched maximal ideal M and quotient field K such that for some element t of M, t has no square root in K. (See [2, p. 248] for such a construction.) If s is a square root of t in an extension field of K, we let $D=V[s]$. D is integral over V; let P be a prime ideal of D lying over M. Since s^2 is in P, s is in P so that $P=MD+(s)$. If Q is P-primary, $Q \cap V$ is M-primary, and hence $Q \cap V=M$. It follows that $MD \subseteq Q \subseteq MD+(s)$ so that $Q=MD$ or $Q=P$. Therefore, the set of P-primary ideals is linearly ordered while MD, the intersection of the set of P-primary ideals, is not prime in D.

4.3. THEOREM. *Suppose P is a branched prime ideal of a domain D such that P-primary ideals are valuation ideals. If $\{Q_\alpha\}$ is the set of P-primary ideals and if some $Q_\beta \neq P$ is such that $Q_\beta W \cap D=Q_\beta$, where W is a valuation ring between D and its quotient field K, then $Q_\alpha W \cap D=Q_\alpha$ for each α.*

Proof. Since $Q_\beta \neq P$, $Q_\beta W \subset PW \subseteq M=\sqrt{(PW)}=\sqrt{(Q_\beta W)}$ so that M is a branched prime ideal of W. If M' is the intersection of the set of M-primary ideals of W, M' is a prime ideal properly contained in M. By choice of M, $P=M \cap D \supset M' \cap D=P'$, P' is prime in D and is the intersection of a collection of P-primary ideals. By Lemma 2.1, P' is the intersection of the collection of all P-primary ideals. Thus W_M is a valuation ring containing W having absolute center P on D and there exist P-primary ideals distinct from P which are contractions of ideals of W_M. Finally, if we show that each Q_α is the contraction of an ideal of W_M, it is clear that each Q_α is also the contraction of an ideal of W. We therefore assume without loss of generality that W has absolute center P on D.

We let w be a valuation associated with W. Then PD_P-primary ideals of D_P are valuation ideals, $Q_\beta D_P \neq PD_P$ is PD_P-primary and is a w-ideal, and if each $Q_\alpha D_P$ is a w-ideal, each Q_α is also a w-ideal [2, pp. 239–240]. Hence we may further assume that D is quasi-local and P is maximal in D.

Now under the assumptions that D is quasi-local with maximal ideal P and that W has absolute center P on D, we prove Theorem 4.3. We denote by $\{C_\beta\}$ the set of P-primary ideals which are w-ideals. As previously observed, $\bigcap_\beta C_\beta=P'=\bigcap_\alpha Q_\alpha$. We wish to show $\{C_\beta\}=\{Q_\alpha\}$—that is, we wish to prove that if Q is P-primary, then $QW \cap D \subseteq Q$. Since $Q \nsubseteq \bigcap_\beta C_\beta$, the linear ordering of the set $\{Q_\alpha\}$ implies $C_\beta \subset Q$

for some β. We choose $x=q\xi \in QW \cap D$ where $q \in Q$, $\xi \in W$. We show $x \in Q$. The ideals $C_\beta+(x)$ and $C_\beta+(q)$ are P-primary. Hence either $C_\beta+(x)\subseteq C_\beta+(q)\subseteq Q$ and $x \in Q$, or $C_\beta+(q)\subseteq C_\beta+(x)$. If $q \in C_\beta$ we have $x \in C_\beta W \cap D=C_\beta\subseteq Q$. And if $q \notin C_\beta$, then $q=r+sx=r+sq\xi$ for some $r \in C_\beta$, $s \in D$. Thus $q(1-s\xi)=r \in C_\beta$. Because $q \notin C_\beta$ and C_β is a w-ideal, it follows that $w(1-s\xi)>0$. Therefore we must have $w(s\xi)=w(s)+w(\xi)=w(1)=0$. But $w(s)$ and $w(\xi)$ are nonnegative; consequently $w(s)=w(\xi)=0$. Therefore s is a unit of W, and hence is a unit of D. Thus $x=s^{-1}(q-r)\in C_\beta+(q)\subseteq Q$. Q.E.D.

We can prove the following result, which is related to Theorem 4.3. All the ideas of the proof are included in previous results, so we shall not supply a proof.

If P-primary ideals of the domain D are valuation ideals and if P_1 is a prime ideal of D contained in each P-primary ideal Q_α, then given any valuation ring V centered on P_1, V contains a valuation ring V_1 containing D such that each P-primary ideal is the contraction of an ideal of V_1.

If P is a branched prime ideal of a domain D such that P-primary ideals are valuation ideals, the proof of Theorem 4.3 establishes the existence of a valuation ring V between D and its quotient field such that V has center P on D and such that each P-primary ideal is the contraction of an ideal of V. In case P is maximal in D and the intersection of the set of P-primary ideals is zero, we obtain the uniqueness of V in Theorem 4.4.

4.4. Theorem. *Suppose P is a maximal ideal of the domain D such that P-primary ideals are valuation ideals and such that the intersection of all P-primary ideals is* (0). *If $\{Q_\alpha\}$ is the set of P-primary ideals, there is a unique valuation ring V between D and its quotient field K such that $Q_\beta V \cap D=Q_\beta$ for some $Q_\beta\neq P$. Moreover, this valuation ring has rank one.*

Proof. Consider some $Q_\gamma\neq P$. As noted previously, there is a valuation ring W with absolute center P on D such that $Q_\gamma W \cap D=Q_\gamma$. $(Q_\gamma W)^{1/2}=(PW)^{1/2}=M$ is necessarily the maximal ideal of W and M is branched. If M' is the intersection of the set of M-primary ideals, then $M' \cap D$ is the intersection of all P-primary ideals—that is, $M' \cap D=(0)$. Because W is an overring of D, this implies $M'=(0)$—that is, W has rank one.

Now suppose V_1 is any valuation ring between D and K such that $Q_\beta V_1 \cap V=Q_\beta$ for some $Q_\beta\neq P$. By Theorem 4.3, $Q_\alpha V_1 \cap V=Q_\alpha$ for each α. For $x \in D$ we define $C_x=xV_1 \cap D$. If $x \in P-\{0\}$, there exists Q_α such that $x \notin Q_\alpha=Q_\alpha V_1 \cap D$. Therefore $x \notin Q_\alpha V_1$, $Q_\alpha V_1\subseteq xV_1$, and $Q_\alpha\subseteq C_x$. This shows that C_x is P-primary and consequently $B_x\subseteq C_x$ where, as in §2, B_y is defined for y in D as $\bigcap_\alpha (Q_\alpha+(y))$. However $xV_1\subseteq B_x V_1$ implies $C_x\subseteq B_x V_1 \cap D=B_x$. Therefore $B_x=C_x$ for any $x \in D$.

We consider now any nonzero element a/b of K. If v_1 is a valuation associated with V_1, $v_1(a/b)\geqq 0$ if and only if $aV_1\subseteq bV_1$. But since C_a and C_b are v_1-ideals, $aV_1\subseteq bV_1$ if and only if $C_a\subseteq C_b$—that is, if and only if $B_a\subseteq B_b$. This shows that V_1 is uniquely determined by D. Hence $V_1=W$ and V_1 has rank one.

Example 5.2 of §5 shows that the assumption $\bigcap_\alpha Q_\alpha=(0)$ in Theorem 4.4 cannot be dropped.

4.5. PROPOSITION. *Let D and P be as in Theorem 4.4. The rank one valuation ring V whose existence is established in Theorem 4.4 is discrete if and only if $P\supset P^2$.*

Proof. We suppose $P\supset P^2$. Then for $n>1$, P^n is a valuation ideal which is not prime. By Corollary 1.4 of [1], P^n is not idempotent. In particular, $P^n\supset P^{n+1}$. We conclude that the powers of P properly descend. By Proposition 2.6 $\{P^n\}_{n=1}^\infty$ is the set of P-primary ideals of D. Thus if $p\in P-P^2$ and if a/b is a nonzero nonunit of V—that is, if $B_a=P^r\subset B_b=P^s$, $B_a\subseteq P^{s+1}=B_bB_p=B_{bp}$ so that $a/bp\in V$. Therefore $a/b\in pV$, pV is the maximal ideal of V, and hence V is discrete.

Conversely, if V is discrete with maximal ideal M, then $\bigcap_{n=1}^\infty (M^n\cap D)=(0)$ so that $M^n\cap D\subset P$ for some n. But $P^n\subseteq M^n\cap D$, and therefore $P^n\subset P$. In particular, $P\supset P^2$.

Example 5.8 will show that even in the case when D is quasi-local with maximal ideal M, primary ideals of D are valuation ideals, $M\supset M^2$, and $\bigcap_{n=1}^\infty M^n=(0)$, D need not be a valuation ring.

5. S-**domains.** We conclude by considering some questions raised in [1], where it was shown that in a weak S-domain, and hence in an S-domain, primary ideals are valuation ideals. The questions of whether a domain in which primary ideals are valuation ideals need be an S-domain or a weak S-domain were left open in [1]. We show in Example 5.8 that $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$ does not imply D is a weak S-domain. Example 5.9 shows that a weak S-domain need not be an S-domain. And Example 5.2 shows that no natural generalization of Theorem 4.4 is valid. Section 5 ends with some questions concerning domains in which primary ideals are valuation ideals which we are unable to answer.

5.1. PROPOSITION. *Let D be a domain with quotient field F such that primary ideals of D are valuation ideals, let K be a field containing F, and let W be a valuation ring with unbranched maximal ideal M such that $W=K+M$. If $D_1=D+M$, D_1 has the following properties*:

(a) $\mathscr{Q}(D_1)\subseteq\mathscr{V}(D_1)$.

(b) *Each ideal of D_1 containing M is of the form $A+M$ for some ideal A of D. Further, $D_1/(A+M)\simeq D/A$ so A is maximal, prime, or primary, respectively, in D if and only if $A+M$ is maximal, prime, or primary, respectively, in D_1.*

(c) *If Q is a primary ideal of D and if V is any valuation ring between D and F such that $QV\cap D=Q$, then for any extension V_α of V to K, $V_\alpha+M$ is a valuation ring and $(Q+M)(V_\alpha+M)\cap D_1=Q+M$.*

Proof. We first prove (b). To show any ideal of D_1 compares with M under $\subseteq$ it suffices to prove that this is true for any principal ideal xD_1. For $x\in M$, $xD_1\subseteq M$. If $x\in D_1-M$, x is a unit of W. Hence $m\in M$ implies $m/x\in M$ also. Therefore

$m \in Mx \subseteq D_1 x$. We obtain the other assertions of (b) by considering the canonical homomorphism of D_1 onto $D_1/M \simeq D$.

To prove (c) we apply Lemma 3.3. Under the natural homomorphism ϕ from W onto $W/M \simeq K$, the inverse image of any valuation ring V_α with quotient field K is again a valuation ring, and this inverse image is $V_\alpha + M$. ϕ sends D_1 onto $D_1/M \simeq D$ and sends $V+M$ onto V. Since V_α is an extension of V, each ideal of V is the contraction of an ideal of V_α. In particular QV is the contraction of an ideal of V_α so that Q is also the contraction of an ideal of V_α. Lemma 3.3 then shows that $\phi^{-1}(Q) = Q+M$ is the contraction of an ideal of $\phi^{-1}(V_\alpha) = V_\alpha + M$. Therefore $Q+M$ must be the contraction to $D+M$ of $(Q+M)(V_\alpha+M)$.

To prove (a) we let Q be a P-primary ideal of D_1, $(0) \subset P \subset D_1$. Since primary ideals of D are valuation ideals, (c) and (b) show that if $P \supset M$, Q is a valuation ideal. If $P \subset M$ we choose $a \in M - P$. If $s \in K$, then $sa \in M$ so that $s = sa/a \in (D_1)_P$. Hence $K+M = W$, a valuation ring, is contained in $(D_1)_P$. Thus $(D_1)_P$ is a valuation ring so that Q is a valuation ideal. Finally, we consider the case $P = M$. In this case QM is an ideal of W with radical M. Because M is unbranched, $M = QM \subseteq Q$. Hence Q is a valuation ideal and our proof is complete.

5.2. Example. In the notation of Proposition 5.1 we let $K = F_0(X, Y)$ where X and Y are indeterminates over the field F_0 and we let $D = (F_0[X])_{(X)}$. On the field $K(\{X_n\}_{n=1}^{\infty})$, $\{X_n\}$ a set of indeterminates over K, there is a valuation ring W with unbranched maximal ideal M such that $W = K+M$ [2, p. 248]. We then define $D_1 = D+M$. D is a rank one discrete valuation ring which has infinitely many extensions to the field K. If $N = XD$ is the maximal ideal of D, 5.1 (b) shows D_1 is quasi-local with maximal $N+M$. $N+M$ is branched and 5.1 (c) shows that there are infinitely many valuation rings V_α between D_1 and its quotient field such that each $(N+M)$-primary ideal of D_1 is the contraction of an ideal of V_α for each α. This example shows that in Theorem 4.4 the assumption that $\bigcap_\alpha Q_\alpha = (0)$ is essential.

Before presenting the examples related to S-domains we require several preliminary results.

5.3. Lemma. *For $1 \leqq i \leqq n$ let V_i be a valuation ring with maximal ideal M_i such that V_i has quotient field K. If $V_i \nsubseteq V_j$ for $i \neq j$, then for any i, $1 \leqq i \leqq n$, $\bigcap_{j \neq i} M_j \nsubseteq V_i$.*

Proof. We let v_i be a valuation associated with the valuation ring V_i, $1 \leqq i \leqq n$. The domain $V = V_1 \cap \cdots \cap V_n$ is a semi-quasi-local Prüfer domain with exactly n maximal ideals $N_1 = M_1 \cap V, \ldots, N_n = M_n \cap V$; further, $V_i = V_{N_i}$ for each i [5, p. 54], [3, pp. 182–184]. Therefore we may choose $x \in (\bigcap_{j \neq i} N_j) - N_i$ and $y \in N_i - (\bigcup_{j \neq i} N_j)$. Hence $v_i(y) > 0$, $v_i(x) = 0$, $v_j(y) = 0$, and $v_j(x) > 0$ for each $j \neq i$. Consequently, $x/y \in (\bigcap_{j \neq i} M_j) - V_i$.

5.4. Lemma. *Let A be an ideal of a domain D and let P be a prime ideal of D not containing A. Then $A_{A \cap P} = D_P$.*

Proof. That $A_{A\cap P}\subseteq D_P$ is clear. If $x=d/n\in D_P$ where $d\in D$ and $n\in D-P$, then for $b\in A-P$, $bd\in A$ and $bn\in A-P$. Hence $x=bd/bn\in A_{A\cap P}$. Q.E.D.

5.5. COROLLARY. *Let $\{V_i\}_{i=1}^n$ be a finite family of valuation rings having a common quotient field K such that $V_i\nsubseteq V_j$ for any $i\neq j$. Let M_i be the maximal ideal of V_i, let $V=\bigcap_{i=1}^n V_i$, and let P_1 be a prime ideal of V_1 properly contained in M_1. If $M=\bigcap_{i=1}^n M_i$ and if $P=P_1\cap M_2\cap\cdots\cap M_n$, then $(V_1)_{P_1}=M_{M-P}$.*

Proof. Because V is a Prüfer domain, $(V_1)_{P_1}=V_{P_1\cap V}\supset V_1=V_{M_1\cap V}$; therefore $P_1\cap V\subset M_1\cap V$. Thus $P_1\cap V$ is a nonmaximal prime ideal of V, so that $P_1\cap V \nsupseteq\bigcap_{i=1}^n (M_i\cap V)=\bigcap_{i=1}^n M_i=M$. Therefore $P_1\cap V$ is a prime ideal of V not containing M. By Lemma 5.4, $V_{P_1\cap V}=(V_1)_{P_1}=M_{M\cap P_1\cap V}=M_P\equiv M_{M-P}$.

5.6. COROLLARY. *Let $\{V_i\}_{i=1}^n$ be a finite family of valuation rings having a common quotient field K such that $V_i\nsubseteq V_j$ for $i\neq j$ and such that each V_i contains some fixed field F; M_i the maximal ideal of V_i. Let $D=F+M$ where $M=M_1\cap\cdots\cap M_n$. D is a quasi-local domain with maximal ideal M and is not a valuation ring if $n>1$ or if $V_1\neq F+M_1$. If P is a nonmaximal prime ideal of D, $P=P_\alpha\cap D$ for some nonmaximal prime P_α of some V_i, and $D_P=(V_i)_{P_\alpha}$ is a valuation ring.*

Proof. Let $x\in D-M$; $x=a+m$ where $a\in F-\{0\}$, $m\in M$. Then $1/x-1/a=-m/a(a+m)$. Since $a(a+m)$ is a unit of each V_i, $m/a(a+m)\in M$. Hence $1/x\in F+M=D$. Thus M is the unique maximal ideal of D. If $m\in M_1-M_2$, then neither m nor $1/m$ is in D so D is not a valuation ring.

If P is a nonmaximal prime of D, P is the center of some valuation ring V between D and K. Since $V\supseteq M_1\cap\cdots\cap M_n=M$, $V\supseteq V_i$ for some i or $V\subseteq V_i$ for some i. The maximal ideal of V_i contracts to M on D and is therefore not contained in the maximal ideal of V. Therefore $V_i\subset V$ so that $V=(V_i)_{P_\alpha}$ for some prime P_α of V_i. Further, P_α is the maximal ideal of V and $P_\alpha\cap D=P\subset M_i\cap D=M$. It follows that $P_\alpha\subset M_i$. By Corollary 5.5, $(V_i)_{P_\alpha}=M_{P_\alpha\cap M}$. But clearly $M_{P_\alpha\cap M}\subseteq D_{P_\alpha\cap D}=D_P$. The reverse containment is clear. Q.E.D.

5.7. PROPOSITION. *Let V_1 and V_2 be distinct valuation rings with a common quotient field K such that $V_2=F+M_2$ for some field F contained in V_1 and V_2, where M_i is the maximal ideal of V_i. If M_1 is unbranched, then primary ideals of the domain $D=F+(M_1\cap M_2)$ are valuation ideals.*

Proof. By Corollary 5.6, D is quasi-local with maximal ideal $M=M_1\cap M_2$. Further if P is a nonmaximal prime of D, D_P is a valuation ring. Hence if Q is a primary ideal of D such that $\sqrt{Q}\neq M$, Q is a valuation ideal.

Therefore we show that if Q is M-primary, Q is a valuation ideal. Let v_i be a valuation associated with the valuation ring V_i. We show that Q is a v_2-ideal. To do this we need to show that if $x\in D$ and if $v_2(x)\geqq v_2(q)$ for some $q\in Q$, then $x\in Q$ [7, p. 340]. We first consider the case $v_2(x)>v_2(q)$. Then if $v_1(x)>v_1(q)$, $x/q\in M$ and $x\in qM\subseteq Q$. If $v_1(x)\leqq v_1(q)$, we first observe that $v_1(x)>0$. This is true since

$x \in D$ implies $x=y+m$ for some $y \in F$, $m \in M_1 \cap M_2$. Because $v_2(x) \geqq v_2(q) > 0$ it follows that $y=0$. Hence $x \in M_1 \cap M_2$ and $v_1(x) > 0$. Because M_1 is unbranched, $(xV_1)^{1/2} = P_1 \subset M_1$. By Corollary 5.5, $(V_1)_{P_1} = M_{M-(P_1 \cap M)}$. In particular, $P_1 \cap M = P_1 \cap M_2 \subset M$. We choose $s \in M - P_1$. It follows that $v_1(s^k) < v_1(x)$ for each positive integer k. Because Q has radical M, $s^n \in Q$ for some positive integer n. The elements $q + s^n$ and $q + s^{n+1}$ are in Q and one of these has v_2-value $\leqq v_2(q)$. If $v_2(s^n) \neq v_2(q)$, $q+s^n$ has this property. And if $v_2(s^n) = v_2(q)$, $v_2(s^{n+1}) \neq v_2(q)$ so $q+s^{n+1}$ has the desired property. Thus there is an integer k such that $u = q+s^k \in Q$ and $v_2(u) \leqq v_2(q)$. It follows that $v_1(u) = v_1(s^k) < v_1(x)$ and $v_2(u) \leqq v_2(q) < v_2(x)$. As previously shown, this implies that $x \in uM \subseteq Q$.

On the other hand, if $v_2(x) = v_2(q)$, then $v_2(x/q) = 0$ and $x/q = r+m$ for some nonzero r in F, $m \in M_2$. Then $x - rq = qm$. Since $m \in M_2$, $v_2(qm) > v_2(q)$. But $x - rq \in M_1$ since $x, q \in M_1$. Thus $qm \in D$, $v_2(qm) > v_2(q)$. We have already established that qm then belongs to Q. As $q \in Q$ and $r \in F \subseteq D$, it follows that $x = rq + qm \in Q$. This completes the proof.

We remark that Proposition 5.7 will generalize to the case of n valuation rings $V_1, V_2, \ldots, V_n$ each containing a fixed field F such that at least $n-1$ of the M_i's are unbranched, and such that there are no containment relations among the V_i's.

5.8. Example. We denote by A the field of algebraic numbers and by $A((X))$ the quotient field of the domain of formal power series in one indeterminate X over A. The field $A((X))$ is uncountable while A is countable so that $A((X))$ has infinite transcendence degree over A. We denote by B a transcendence basis of $A((X))$ over A such that $1/X \in B$. Using the same construction that is outlined in [2, p. 248] we obtain a valuation ring V with maximal ideal M such that $A(B)$ is the quotient field of V, $B \subseteq M$, and $V = A + M$. V has an extension to a valuation ring V_1 with quotient field $A((X))$. If M_1 is the maximal ideal of V_1, $A \simeq V/M$ is isomorphic to a subfield of V_1/M_1. Because $A((X))$ is algebraic over $A(B)$, V_1/M_1 is algebraic over A. Since A is algebraically closed we must have $A = V_1/M_1$. However, $A \subseteq V_1$ and $A \cap M = (0)$. Consequently, $V_1 = A + M_1$. Finally, $A((X))$ algebraic over $A(B)$ implies that the ordinal type of the set of prime ideals of V is the same as the ordinal type of the set of prime ideals of V_1. Hence the maximal ideal of V_1 is also unbranched. Finally, for V_2 we take the rank one discrete valuation ring $V_2 = A[[X]] = A + M_2$ where $M_2 = XV_2$ is the maximal ideal of V_2.

By Proposition 5.7, primary ideals of the domain $D = A + (M_1 \cap M_2)$ are valuation ideals. Also, D is quasi-local with maximal ideal $M = M_1 \cap M_2$ and $M \neq M^2$. By Proposition 2.6, $\{M^n\}_{n=1}^{\infty}$ is the set of M-primary ideals. Since $M \subseteq M_2$, $\bigcap_{n=1}^{\infty} M^n = (0)$ in this case. Hence D is not a weak S-domain. There is, in fact, an uncountable chain of prime ideals properly between M and (0).

5.9. Example. Let $\{X_i\}_{i=1}^{\infty}$ be a countable collection of indeterminates over a field K and for each i we set $Y_i = 1/X_i$. We define valuations v_1 and v_2 on $K(\{X_i\}_1^{\infty})$ by defining them on $K[\{X_i\}]$ and $K[\{Y_i\}]$ as follows, then taking the canonical extension in each case to $K(\{X\})$. v_1 is defined on $K[\{X_i\}]$ and has value group G_1, the countable weak direct sum of the additive group Z of integers, ordered lexicographically; v_2 is defined on $K[\{Y_i\}]$ and has value group G_2, the countable weak direct sum of Z with the reverse lexicographic ordering. v_1 is defined on a nonzero monomial $aX_1^{e_1}\cdots X_n^{e_n}$ to be $(e_1,\ldots,e_n,0,\ldots)$, and $v_1(f)$, for a nonzero element f of $K[\{X_i\}]$ as the minimum of the v_1-values of the nonzero monomials in f. v_2 is similarly defined on $K[\{Y_i\}]$ with the role of the X_i's played by the Y_i's. If V_i is the valuation ring of v_i, $V_i=K+M_i$ where M_i is the maximal ideal of V_i; also, V_1 and V_2 are independent and M_1 is unbranched. Hence primary ideals of

$$D = K+(M_1 \cap M_2)$$

are valuation ideals. The ideal M_2 is branched. If P_2 is the intersection of the set of M_2-primary ideals, then $M_1 \cap P_2$ is the intersection of all $(M_1 \cap M_2)$-primary ideals of D and there are no prime ideals of D properly between $M_1 \cap M_2$ and $M_1 \cap P_2$. Yet there is a countable chain of prime ideals of D distinct from $M_1 \cap M_2$ which are not contained in $M_1 \cap P_2$. It follows that D is a weak S-domain but not a strong S-domain.

We conclude by listing some questions concerning primary ideals and valuation ideals which we are unable to answer. Where they are meaningful we give the local form of the question. We have not been able to settle these questions in the local or global case. P denotes a branched prime ideal of a domain D.

1. Suppose each P-primary ideal is a valuation ideal, that $Q\neq P$ is P-primary, and that P_1 is a prime ideal contained in Q.

(a) Are P/P_1-primary ideals of D/P_1 valuation ideals?

(b) Is P_1 contained in each P-primary ideal?

2. Suppose $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$ and D_1 is the integral closure of D. Are primary ideals of D_1 valuation ideals?

We remark that questions 1(a) and 1(b) are equivalent. If D_1 is an arbitrary overring of D integral over D, $\mathscr{Q}(D)\subseteq\mathscr{V}(D)$ does not imply $\mathscr{Q}(D_1)\subseteq\mathscr{V}(D_1)$.

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